Definitions regarding (for $f: D \to \mathbb{R}$), $l \in \mathbb{R}$

$$\lim_{x \to x_0} f(x) = \begin{cases} l \\ +\infty \\ -\infty \end{cases} \quad (x_0, \text{ cluster w.r.t. } D)$$

$$\lim_{x \to x_0+} f(x) = \begin{cases} \dots \\ \dots \\ \dots \end{cases} \quad (\text{right-hand limit, } x_0 \text{ cluster w.r.t. } D \cap (x_0, +\infty))$$

$$\lim_{x \to x_0-} f(x) \quad (\text{left-hand } \dots)$$

$$\lim_{x \to +\infty} f(x) = \begin{cases} l \\ +\infty \\ -\infty \end{cases} \quad (+\infty, \text{ "cluster" w.r.t. } D: \ \forall a \in \mathbb{R}\ \exists x \in D \text{ s.t. } x \geq a)$$

$$\lim_{x \to -\infty} f(x) = \begin{cases} l \\ +\infty \\ -\infty \end{cases} \quad (\forall a \in \mathbb{R}\ \exists x \in D \text{ s.t. } x \leq a)$$

Theorems: Uniqueness, but no "local boundedness result"
careful with computation rules.

Examples

1 $\lim_{x \to 0+} \frac{1}{x} = +\infty$, $\lim_{x \to 0-} \frac{1}{x} = -\infty$, $\lim_{x \to 0} \frac{1}{x}$ not exist.

The last assertion follows from the following theorem:

Theorem (on one-sided limits). Let $\lim_{x \to x_0-} f(x) = l_1$ and $\lim_{x \to x_0+} f(x) = l_2$ where $l_1, l_2 \in [-\infty, \infty]$. If $l_1 = l_2$ then $\lim_{x \to x_0} f(x)$ exists and equals $l_1 = l_2$.

Conversely, if $\underbrace{\lim_{x \to x_0} f(x)}^{=l}$ exists in $[-\infty, \infty]$, then $\lim_{x \to x_0-} f(x) = l = \lim_{x \to x_0+} f(x)$.

1: $\lim\limits_{x \to 3} \dfrac{x+1}{x-2} = 4$


$$\left|\frac{x+1}{x-2} - 4\right| = \left|\frac{-3x+9}{x-2}\right| = \left|\frac{-3(x-3)}{x-2}\right| \leq \frac{3|x-3|}{|x-2|} \quad (x \neq 2)$$

(1) Why the following answer is wrong? Let $\varepsilon > 0$
Take $\delta = \frac{\varepsilon}{3}|x-2|$. Then if $|x-3| < \delta$ then

$$\left|\frac{x+1}{x-2} - 4\right| = \cdots \leq \frac{3|x-3|}{|x-2|} < \frac{3 \cdot \frac{\varepsilon}{3}|x-2|}{|x-2|} = \varepsilon.$$

(what happen $x'$ is different from $x$ and $0<|x'-2|<\delta$ ?)

(2) How to think for a solution. Let $\varepsilon > 0$ In attempting to chose a good $\delta > 0$, one has to ensure that ~~for all x ∈~~ $V_\delta(3) = (3-\delta, 3+\delta)$ is of ~~dis~~ (strictly) positive distance from 2 (so as the factor $|x-2|$ (appeared in the denominator) is bounded below by a positive constant $c > 0$, $\forall x \in V_\delta(3)$ ~~)~~;

$$|x-2| \geq c \quad \text{i.e.} \quad \frac{1}{|x-2|} \leq \frac{1}{c}$$

$$\forall x \in V_\delta(3).$$

Now, with $\delta > 0$,

$$\text{dist}(2, V_\delta(3)) = \begin{cases} 0 & \text{if } \delta \geq 1 \\ 1-\delta & \text{if } \delta < 1 \end{cases}$$

we are led to require that $\delta < 1$ (in addition to being positive). Having set this "preliminary condition" one notes that, $\forall x \in V_\delta(3) = (3-\delta, 3+\delta)$, one has

$$|x-2| = x-2 > (3-\delta)-2 = 1-\delta$$

and so

$$\frac{3|x-3|}{|x-2|} \leq \frac{3|x-3|}{1-\delta} < \frac{3\delta}{1-\delta} \leq \varepsilon$$

if $1-\delta \geq c\ (>0)$ and $\delta \leq \frac{c\varepsilon}{3}$. Thus

one can take, e.g. $\delta := \min\left\{\frac{1}{2}, \frac{\varepsilon}{6}\right\}$ or, for another choices

$$\delta := \min\left\{\frac{1}{3}, \frac{2\varepsilon}{9}\right\}$$

$$\delta := \min\left\{\frac{2}{3}, \frac{\varepsilon}{9}\right\} \text{ etc.}$$

(3) <u>Formal Solution</u>. Let $\varepsilon > 0$. Let $\delta$ be defined by

$$\delta := \min\left\{\frac{1}{2}, \frac{\varepsilon}{6}\right\} \quad (>0)$$

Let $x \in V_\delta(3) = (3-\delta, 3+\delta)$. It suffices to show that

$$(*) \quad \left|\frac{x+1}{x-2} - 4\right| < \varepsilon.$$

To do this, note that

$$3-\frac{1}{2} \leq 3-\delta < x < 3+\delta \leq 3+\frac{1}{2}$$

and so

$$\frac{1}{2} < x-2 < \frac{3}{2}.$$

Hence

$$\left|\frac{x+1}{x-2} - 4\right| = \left|\frac{x+1-4(x-2)}{x-2}\right| = \left|\frac{-3x+9}{x-2}\right| = \frac{3|x-3|}{|x-2|}$$

$$< \frac{3|x-3|}{1/2} = 6|x-3| < 6\delta \leq 6\cdot\frac{\varepsilon}{6} = \varepsilon,$$

verifying $(*)$.

2: $\lim_{x\to 3} \frac{x^3+1}{x-2} = 28$

Let $\varepsilon > 0$. Let $\delta > 0$ be defined by

$$\delta := \min\left\{\frac{1}{2}, \frac{\varepsilon}{100}\right\}$$

Let $x$ be s.t. $|x-3| < \delta$. It suffices to show that

(*) $\left|\frac{x^3+1}{x-2} - 28\right| < \varepsilon$.

Note that

$$|x-3| < \delta \leq \frac{1}{2} \quad \text{so} \quad \frac{5}{2} < x < 3\frac{1}{2}$$

and hence

$$\frac{1}{2} \leq |x-2| \quad \& \quad |x| < 4$$

Consequently

$$\left|\frac{x^3+1}{x-2} - 28\right| = \frac{|x^3+1-28(x-2)|}{|x-2|} = \frac{|x-3|\,|x^2+3x-19|}{|x-2|}$$

$$\leq \frac{|x-3| \cdot (|x|^2+3|x|+19)}{1/2} \leq 100\,|x-3| < 100\,\delta \leq \varepsilon,$$

verifying (*).